Proceedings of the 5th International Conference on Interdisciplinary Humanities and Communication Studies
DOI: 10.54254/2753-7064/2026.36988

A Method for Propositionalizing Prenex Normal Form Sentences under Inclusive Semantics and Testing the Validity of Inferences Formed from Such Sentences

Peifeng Li

Art & Science College, Syracuse University, Syracuse, USA
pli151@syr.edu

Abstract. Quantifier expansion requires enumerating the objects in the domain one by one. Therefore, it can only handle finite domains. This paper proposes a different perspective on propositionalization: rather than counting objects in the domain, it focuses on the finite regions determined by the sentence. Therefore, within its scope of application, even if the domain contains infinitely many objects, this method can still produce a finite propositionalization. This paper distinguishes two kinds of FOL based on different semantics: basic FOL, which allows the domain to be empty, and strengthened FOL, which requires a non-empty domain. This paper focuses on the basic FOL. At present, the basic FOL method only applies to prenex normal form sentences. To propositionalize such sentences, this paper introduces the Region Table and uses it to translate prenex-normal-form sentences into TFL sentences. On this basis, FOL inferences composed of such sentences can also be translated into TFL inferences, allowing their validity to be tested using TFL rules. This method provides a new approach to propositionalizing some FOL sentences over infinite domains and testing the validity of inferences composed of them, and it may be applied to formal reasoning in mathematics and physics.

Keywords: First-order Logic, Propositionalization, Empty Domain, Prenex Normal Form, Region Table

1. Introduction

Although quantifier expansion provides a mechanical method of propositionalization, it requires listing the objects in the domain one by one. Since a proposition has a finite length, this method only applies to finite domains. This paper proposes a method for propositionalizing prenex normal form sentences in infinite domains.

In addition, Classic FOL also has limitations. It requires a non-empty domain. However, this requirement doesn't hold in all meaningful contexts. An example is set theory. Although sets are the objects of set theory, the cumulative hierarchy of ZFC is also constructed from the empty stage. $V_0 = \phi$. Similar cases can also be found in the history of science. In the 19th century, scientists realized that light is also a wave, so they believed that light, like other waves, has a medium and called it "Aether." Scientists even developed an entire theoretical system for it. Of course, the Aether

© 2026 The Authors. This is an open access article distributed under the terms of the Creative Commons Attribution License 4.0 (https://creativecommons.org/licenses/by/4.0/).

DOI: 10.54254/2753-7064/2026.36988

doesn't exist, and because its role in the theory became less and less important, it gradually faded from history. But who can guarantee that some scientific theories today won't become the new "Aether"? The situation is similar in literature. A novel can describe a completely fictional world. If its domain is taken to be the objects in that fictional world, and none of these objects exists in reality, then the domain is empty. Such examples are countless. Therefore, empty terms and empty domains aren't meaningless, whether considered from the perspective of science, mathematics, or literature. This paper will show how to propositionalize FOL sentences in prenex normal form under inclusive semantics that allow an empty domain. It will systematically translate these FOL sentences into TFL sentences. In this way, some FOL inferences involving such sentences can be tested by TFL rules to determine whether the corresponding rules of inference are valid.

This paper uses $a, b, c, \ldots$ for individual constants; $x, y, z, \ldots$ for individual variables; The logical connectives are denoted by $\neg$, $\vee$, $\wedge$,$\rightarrow$ and $\leftrightarrow$; the quantifiers by $\forall$ and $\exists$; Greek letters such as $\psi, \varphi, \ldots$ are used to denote arbitrary formulas. When it is necessary to show the free individual variables of a formula, it's written as $\psi(x, y, z, \ldots)$. Other symbols used in this paper will be introduced later.

2. Theoretical framework and preliminaries

2.1. Objects, predicates and construction methods

2.1.1. Objects and predicates

Whitehead and Russell state that, "The universe consists of objects having various qualities and standing in various relations" [1]. Aristotle classified qualities and relations into ten categories [2]. These categories can be understood as a classification of predicate expressions [3]. Therefore, the non-logical vocabulary used to describe the world consists roughly of names for objects and predicates [4].

2.1.2. Two construction methods: object-driven and predicate-driven

Consider constructing a model. The model contains objects, and each object belongs to the extensions of the corresponding predicates according to its properties, as shown in Figure 1.

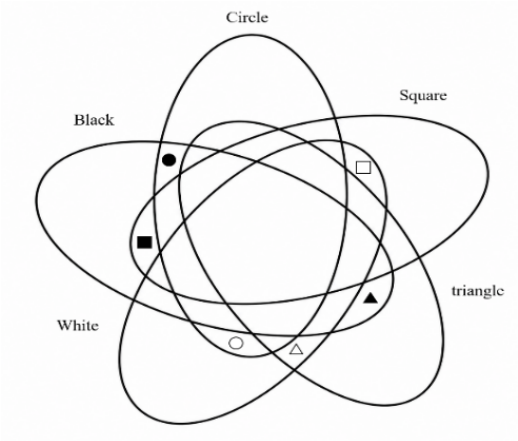


Figure 1. Model

DOI: 10.54254/2753-7064/2026.36988

Readers familiar with the inverted spectrum may think that this example touches on a deeper issue. The inverted spectrum refers to the possibility that a person's color experiences may be different from those of others since their birth, while this difference isn't reflected in their descriptions of color [5]. For example, a non-inverted person may have a red visual experience when seeing an apple, but an inverted person looking at the same apple may have an experience like the non-inverted person's experience of blue. For the inverted person, call the blue red, since they can't distinguish between their descriptions of color since birth. Therefore, color isn't a suitable example for the model. However, this paper doesn't discuss this issue further. The black, white, circular, square, and triangular examples are used only to illustrate that objects can have different properties; they don't concern the philosophical issues of color experience or qualia.

This model can be constructed in two ways: object-driven construction (ODC) and predicate-driven construction (PDC).

1. Object-driven Construction (ODC)

Aristotle thought that perception is a "power of discrimination"; it doesn't require that creatures have concepts, but only intuition [6]. Objects already exist in the world; humans distinguish and classify them through perception, and then give names to these classifications. This process is called object-driven construction (ODC) (Figure 2).

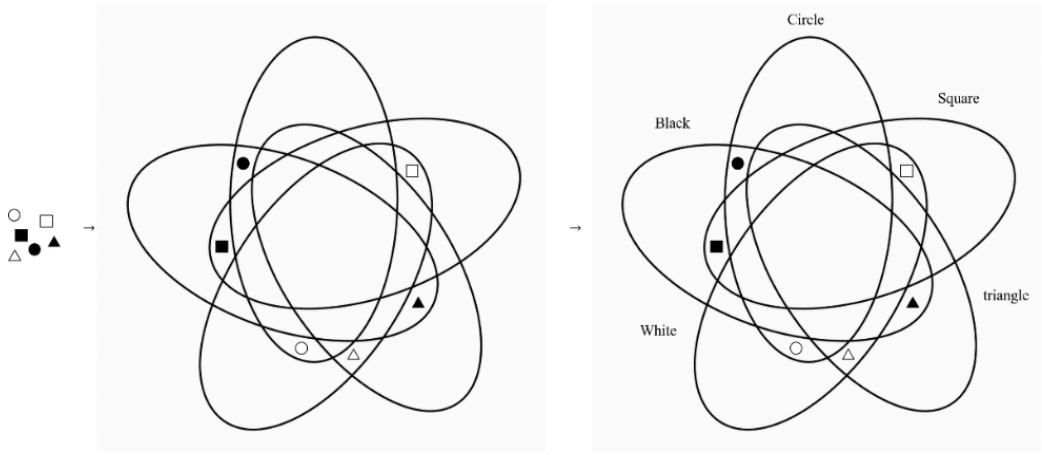


Figure 2. Object-driven construction

However, ODC can't allow empty predicates, because predicates can only be formed by classifying existing objects. Therefore, no empty predicate can exist within ODC. This is where PDC becomes necessary.

2. Predicate-driven Construction (PDC)

PDC base on ODC. New concepts can be formed from existing concepts through introspection, abstraction, extrapolation, and recombination [7]. These concepts may have empty extensions. They are like empty containers, waiting for suitable objects to enter them. As Lao Tzu said, "Clay is fashioned into vessels; but it is on their empty hollowness that their use depends" [8]. PDC begins with empty predicates that are waiting to be filled by objects, ultimately forming the model shown in Figure 3.

DOI: 10.54254/2753-7064/2026.36988

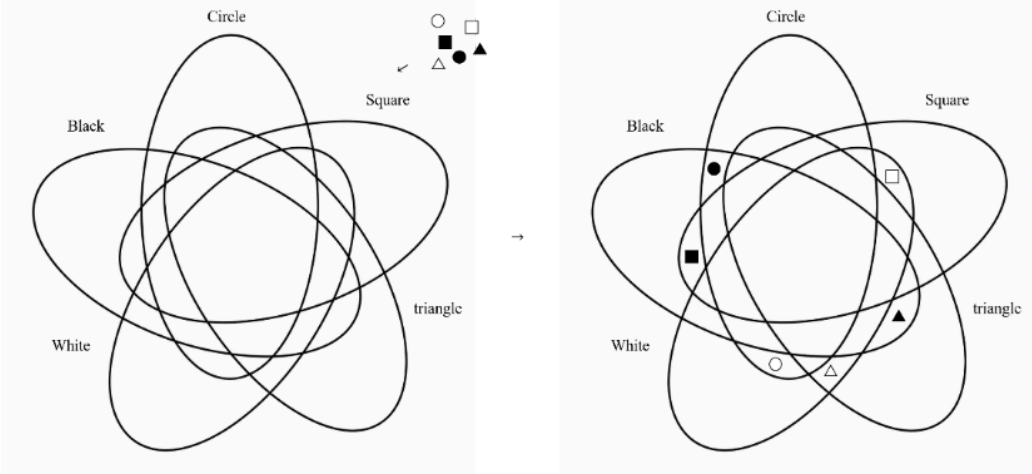


Figure 3. Predicate-driven construction

PDC doesn't assume the Universe is empty, but the extensions and domains of its predicates can be empty.

2.2. Basic syllogism and strengthened syllogism

2.2.1. Validity of basic and strengthened syllogisms

Basic syllogisms are constructed within PDC. "Basic" means do not suppose a non-empty domain. Strengthened syllogisms are based on basic syllogisms but additionally require the subject term to be non-empty. "Strengthened" means to suppose a non-empty domain. The following example will illustrate this in detail.

The definition of the validity of a syllogism is that a syllogism is valid if and only if the Venn diagram representing all the premises contains the Venn diagram representing the conclusion [9]. A Venn diagram divides the domain into several regions using three intersecting circles. These three circles represent the major term, the middle term, and the minor term, respectively. Shading represents an empty region, and an "x" in a region indicates the existence of at least one object in that region.

2.2.2. Examples of basic and strengthened syllogisms

Consider the syllogism: "All mammals are animals; some pets are mammals; therefore, some pets are animals." Let M stand for mammals, A for animals, and P for pets. Figure 4 shows it's valid.

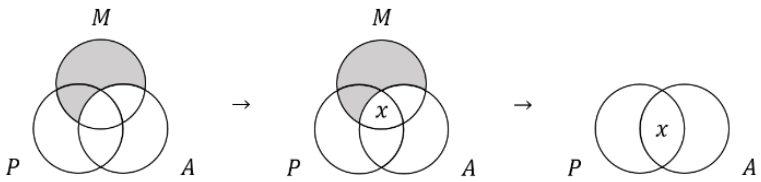


Figure 4. Pet's example

However, some inferences that seem intuitively valid are shown to be invalid by Venn diagrams. For example, "All whales are mammals; all whales are sea animals; therefore, some mammals are

DOI: 10.54254/2753-7064/2026.36988

sea animals." Let W stand for whales, M for mammals, and S for sea animals. Figure 5 shows it's invalid.

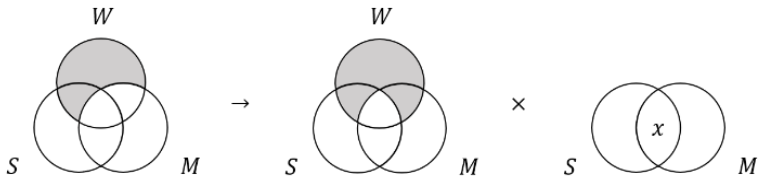


Figure 5. Whale's example in basic syllogisms

This is because the premise "Whales exist" is missing. Consider another example: "All dragons are reptiles; all dragons breathe fire; therefore, some reptiles breathe fire." This inference is intuitively invalid, since dragons don't exist. Thus, adding the premise "Whales exist" makes the previous inference valid (Figure 6).

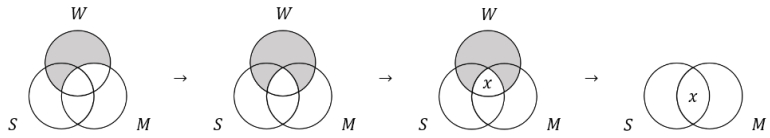


Figure 6. Whale's example in strengthened syllogisms

This distinction between basic and strengthened syllogisms provides the foundational intuition for extending the concepts to First-order Logic. Basic FOL is defined as a semantic framework that permits the domain of discourse to be empty, whereas Strengthened FOL (equivalent to classical FOL) presupposes a non-empty domain. This paper primarily develops the propositionalization method for Basic FOL.

2.3. Basic FOL and strengthened FOL

This paper aims to extend "basic" and "strengthened" to FOL. Basic FOL doesn't require the domain to be non-empty, but strengthened FOL does. Therefore, classical FOL is a special case of strengthened FOL.

Because the propositionalized sentences are designed to be more basic than FOL and set theory, the propositionalized sentences and the language used to explain the propositionalization shouldn't use the language of set theory, except for the most basic membership and set symbols.

3. Region, region value and region table

3.1. Conceptual framework

Venn diagrams represent relations between sets or predicates visually. However, when there are more than three sets or predicates, drawing a Venn diagram becomes very difficult. As shown in Figure 1, even a simple case involving only six elements requires five circles. To address this, this paper introduces the Region Table, a finite, textual representation that partitions the domain based on the region values of sub-formulas within a matrix, which will be mentioned later.

Let d denote the variables for all elements in a domain. Variables such as x, y, and z are all part of d. The objects referred to by any variable can form a set. For example, the set of objects referred to by x is denoted by $\{x\}$.

DOI: 10.54254/2753-7064/2026.36988

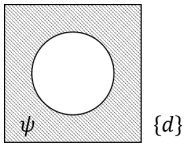
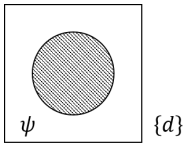
A prenex normal form sentence consists of a quantifier prefix and a matrix. Greek letters such as ψ and φ denote arbitrary subformulas in the matrix. The matrix separates the domain into two parts: the inside and the outside. The inside includes all objects that fit in the matrix. The outside includes all objects that do not fit in it. No object can be both inside and outside the matrix. No object can be inside or outside the matrix. Since, no matter the quantifier in the prefix is $\forall$ or $\exists$, the regions of the matrix are still unchanged, the quantifier prefix doesn't need to be included in the table.

Let I denote the inside of a formula's extension and O the outside.

3.2. Construction of the region table

These tables also have corresponding Venn diagrams. Let the rectangle represent the domain, the dotted region be the outside of the matrix, and the blank region be the inside of the matrix. This gives the following Table 1.

Table 1. Region table of (ψ) and $(\neg\psi)$

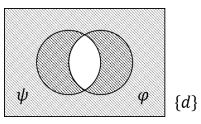
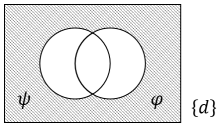
	
(ψ)	$(\neg\psi)$
I	O
O	I

This means that the inside of the matrix (ψ) is the outside of the matrix $(\neg\psi)$, and the outside of (ψ) is the inside of $(\neg\psi)$.

Such a table is called a Region Table, and I and O are called Region Values. The outermost parentheses of a formula indicate that the whole formula is the matrix of a prenex normal form sentence.

3.3. Extension to connectives

Table 2. Region Table of $(\psi \wedge \varphi)$ and $(\psi \vee \varphi)$

			
(ψ)	(φ)	$(\psi \wedge \varphi)$	$(\psi \vee \varphi)$

DOI: 10.54254/2753-7064/2026.36988

Table 2. (continued)

I	I	I	I
I	O	O	O
O	I	O	I
O	O	O	I

Table 2 shows that a region inside both matrices (ψ) and (φ) is inside the matrix $(\psi \wedge \varphi)$, while all other regions are outside of it. A region outside both matrices (ψ) and (φ) is outside the matrix $(\psi \vee \varphi)$, while all other regions are inside it.

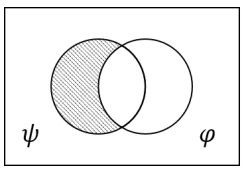
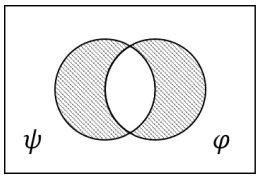
Writing in this way is cumbersome. It is much simpler to name each region according to its Region Values. Let I_ψ indicate that a region is inside ψ, and O_ψ indicate that a region is outside ψ ; the same notation applies to φ. Then, the four regions can be named as $(I_\psi,\ I_\varphi)$, (I_ψ, O_φ), (O_ψ, I_φ), and (O_ψ, O_φ). If an order is fixed in advance, with the Region Value of ψ first and of φ second, the subscripts can be omitted, giving (I, I), (I, O), (O, I), and (O, O). Before a proof, the order can be specified as $(val\ \psi, val\ \varphi)$. The same order will be used throughout the following discussion. This method applies not only to a matrix with two formulas but also to one with more.

Each region could also be named using set-theoretic notation. The four regions could be written as $\psi \cap \varphi$, $\psi \bigcap \overline{\varphi}$, $\overline{\psi} \bigcap \varphi$, and $\overline{\psi} \bigcap \overline{\varphi}$. However, since the language of set theory isn't allowed at this stage, this notation can't be used here.

3.4. Implication and biconditional

Next, the remaining two are added.

Table 3. Region Table of $(\psi \rightarrow \varphi)$ and $(\psi \leftrightarrow \varphi)$

		 $\{d\}$	 $\{d\}$
(ψ)	(φ)	$(\psi \rightarrow \varphi)$	$(\psi \leftrightarrow \varphi)$
I	I	I	I
I	O	O	O

DOI: 10.54254/2753-7064/2026.36988

Table 3. (continued)

O	I	I	O
O	O	I	I

Table 3 shows that (I,O) is outside the matrix $(\psi \to \varphi)$, while all other regions are inside it. It also shows that (I, I) and (O, O) are inside the matrix $(\psi \leftrightarrow \varphi)$, while the other two regions are outside it.

No matter how the quantifiers change, the regions determined by the subformulas in the matrix remain the same. Their corresponding names also remain the same. Since these regions do not change and are independent of the quantifiers, there is no need to enumerate them one by one. Moreover, under inclusive semantics, the objects in the domain cannot always be enumerated one by one. It is only necessary to consider whether each region contains any object.

Since every FOL sentence has a finite length, it contains only finitely many predicates. Therefore, these predicates divide the domain into only finitely many regions. Whether the extension of a predicate is finite or infinite does not affect the number of regions. By enumerating the possible status of each region, an FOL sentence under an infinite domain can be expressed by finite TFL sentences. This is the main point of this paper.

4. Propositionalization of basic FOL

4.1. Propositionalization of monadic basic FOL sentences

If a Region, for example (I, I), contains objects, and these objects belong to $\{x\}$, this can be written as an atomic proposition $\exists x(I, I)$, read as "there is an x in (I, I)."

Prenex normal form sentences can be propositionalized according to the following table. Take $(\psi \leftrightarrow \varphi)$ for example, according to its region table, (I, I) and (O, O) are inside the formula, while (I, O) and (O, I) are outside it. For $\exists x(\psi \leftrightarrow \varphi)$, there is at least one object in (I, I) or (O, O), but it is unknown which of the two regions contains it. Therefore, it's equivalent to $\exists x(I, I) \vee \exists x(O, O)$. For $\forall x(\psi(x) \leftrightarrow \varphi(x))$, under inclusive logic, it can't be determined whether there are any objects inside the sentence, but it can be determined that there are no objects outside it. After propositionalization, it should be

$$(\neg\exists x(I, O) \wedge \neg\exists x(O, I)) \wedge ((\exists x(I, I) \vee \exists x(O, O)) \vee \neg(\exists x(I, I) \vee \exists x(O, O))) \quad (1)$$

It is therefore equivalent to $\neg\exists x(I, O) \wedge \neg\exists x(O, I)$. All other cases can be handled in the same way. This is the method for propositionalizing monadic FOL sentences.

4.2. Proofs examples for monadic basic FOL inference

Take modus ponens in monadic basic FOL as an example,

DOI: 10.54254/2753-7064/2026.36988

$$\forall x(\psi(x) \to \varphi(x)), \exists x(\psi(x)) \vdash \exists x(\varphi(x)) \tag{2}$$

Table 4. Region table of $(\psi(x))$, $(\varphi(x))$, and $\forall x(\psi(x) \to \varphi(x))$

$(\psi(x))$	$(\varphi(x))$	$\forall x(\psi(x) \to \varphi(x))$
I	I	I
I	O	O
O	I	I
O	O	I

Fix the order of the Region Values as $(val\psi(x), val\varphi(x))$. According to Table 4,

$$\forall x(\psi(x) \to \varphi(x)) \leftrightarrow \neg\exists x(I, O) \tag{3}$$

$$\exists x(\psi(x)) \leftrightarrow \exists x(I, I) \vee \exists x(I, O) \tag{4}$$

$$\exists x(\varphi(x)) \leftrightarrow \exists x(I, I) \vee \exists x(O, I) \tag{5}$$

Thus, the proof is as follows in Table 5:

Table 5. Proof of (2)

Number	Proof	Law
1	$\forall x(\psi(x) \to \varphi(x))$	Pr
2	$\exists x(\psi(x))$	Pr
3	$\neg\exists x(I, O)$	$\leftrightarrow$ E , (3), 1

DOI: 10.54254/2753-7064/2026.36988

Table 5. (continued)

4	$\exists x(I,I) \vee \exists x(I,O)$	$\leftrightarrow$ E , (4), 2
5	$\exists x(I,I)$	*DS, 4, 3,*
6	$\exists x(I,I) \vee \exists x(O,I)$	$\vee$I , 5
7	$\exists x(\varphi(x))$	$\leftrightarrow$ E , (5), 6

Therefore, this instance of Modus Ponens is valid.

4.3. Propositionalization of polyadic basic FOL sentences

Since all variables in polyadic basic FOL sentences range over the same domain $\{d\}$, the propositionalization method for monadic basic FOL can still be applied to each variable. For example, $\forall x \exists y(\psi(x,y))$ can first be propositionalized as $\neg\exists x(O_\psi) \wedge \exists y(I_\psi)$. However, it's not enough, because the sentence also contains relations between the variables.

Boole proposed the concept of "selection,", which means to select objects that satisfy the corresponding predicate from objects through predication [10]. This paper adopts this concept to express relations: when one object is selected, objects that have the corresponding relation to it are also selected accordingly. For example, $\mathscr{S}x(I,I)$ is means " x is selected in (I,I)," and $\neg\mathscr{S}x(I,I)$ is used to mean " x is not selected in (I,I)."

Take $\forall x \exists y(\psi(x,y))$ for example. Since whether there is any object inside $\psi(x,y)$ is uncertain, this study can't directly write $\mathscr{S}x(I_\psi)$. However, there is no object belonging to $\{x\}$ outside $\psi(x,y)$, so no x can be selected from (O_ψ). Thus, this study can write $\neg\mathscr{S}x(O_\psi) \wedge (\mathscr{S}x(I_\psi) \vee \neg\mathscr{S}x(I_\psi))$, which is equivalent to $\neg\mathscr{S}x(O_\psi)$. Therefore, $\forall x \exists y(\psi(x,y))$ implies the following relation: if x isn't selected from (O_ψ), then y is selected from (I_ψ), namely, $\neg\mathscr{S}x(O_\psi) \rightarrow \mathscr{S}y(I_\psi)$. Thus,

$$\forall x \exists y(\psi(x,y)) \leftrightarrow ((\neg\exists x(O_\psi) \wedge \exists y(I_\psi)) \wedge (\neg\mathscr{S}x(O_\psi) \rightarrow \mathscr{S}y(I_\psi))) \tag{6}$$

When there are more than two variables, the propositionalization of the "selection" part can be written in a nested structure similar to the following form:

$$\neg Sx(O_\psi) \rightarrow (\mathscr{S}y(I_\psi) \rightarrow (\mathscr{S}z(I_\psi) \rightarrow \mathscr{S}u(I_\psi))) \tag{7}$$

If a premise in a polyadic inference is a monadic basic FOL sentence, the original meaning of the concept of "selection" is retained. For example, the meaning of includes an instruction, namely, " x is selected in (I_φ)." Therefore,

DOI: 10.54254/2753-7064/2026.36988

$$\exists x(\varphi(x)) \leftrightarrow (\exists x(I_\varphi) \wedge \mathscr{S}x(I_\varphi)) \tag{8}$$

$$\forall x(\varphi(x)) \leftrightarrow (\neg\exists x(O_\varphi) \wedge \neg\mathscr{S}x(O_\varphi)) \tag{9}$$

4.4. Proof examples for polyadic basic FOL laws

Take Modus Ponens in polyadic basic FOL,

$$\forall x\exists y(\psi(x,y) \to \varphi(x,y)),\ \forall x\forall y(\psi(x,y)) \vdash \forall x\exists y(\varphi(x,y)) \tag{10}$$

as an example. Let the order of the Region Values be $(val\ \psi(x), val\ \varphi(y))$. According to Table 4,

$$\forall x\exists y(\psi(x,y) \to \varphi(x,y)) \leftrightarrow (\neg\exists x(I,O) \wedge (\exists y(I,I) \vee \exists y(O,I) \vee \exists y(O,O))) \tag{11}$$
$$\wedge((\neg Sx(I,O)) \to (Sy(I,I) \vee Sy(O,I) \vee Sy(O,O)))$$

$$\forall x\forall y(\psi(x,y)) \leftrightarrow ((\neg\exists x(O,I) \wedge \neg\exists x(O,O)) \wedge (\neg\exists y(O,I) \wedge \neg\exists y(O,O))) \tag{12}$$

$$\wedge((\neg\mathscr{S}x(O,I) \wedge \neg\mathscr{S}x(O,O)) \to (\neg\mathscr{S}y(O,I) \wedge \neg\mathscr{S}y(O,O)))$$

$$\forall x\exists y(\varphi(x,y)) \leftrightarrow ((\neg\exists x(I,O) \wedge \neg\exists x(O,O)) \wedge (\exists y(I,I) \vee \exists y(O,I))) \tag{13}$$

$$\wedge((\neg Sx(I,O) \wedge \neg Sx(O,O)) \to (Sy(I,I) \vee Sy(O,I)))$$

Thus, the proof is as table 6, which shows (10) is valid.

4.5. Inferences valid in classical FOL but invalid in basic FOL

Some inferences, such as

$$\forall x\exists y(\psi(x) \to \varphi(y)), \forall x(\psi(x)) \vdash \exists y(\varphi(y)) \tag{14}$$

Classical FOL can prove them to be valid in Table 6.

DOI: 10.54254/2753-7064/2026.36988

Table 6. Proof of (14) in classic FOL

Number	Proof	Law
1	$\forall x \exists y(\psi(x) \to \varphi(y))$	Pr
2	$\forall x(\psi(x))$	Pr
3	$\exists y(\psi(a) \to \varphi(y))$	$\forall$E , 1
4	$\psi(a)$	$\forall$E , 2
5	$\psi(a) \to \varphi(b)$	Assumption (flag b)
6	$\varphi(b)$	$\to$ E , 5, 4
7	$\exists y(\varphi(y))$	$\exists$I , m
8	$\exists y(\varphi(y))$	$\exists$E , 3, 5–8

However, in basic FOL, although the premises are consistent, the inference is invalid. According to Table 7,

$$\forall x \exists y(\psi(x) \to \varphi(y)) \leftrightarrow ((\neg \exists x(I, O)) \wedge (\exists y(I, I) \vee \exists y(O, I) \vee \exists y(O, O))) \tag{15}$$

$$\wedge((\neg \mathscr{S} x(I, O)) \to (\mathscr{S} y(I, I) \vee \mathscr{S} y(O, I) \vee \mathscr{S} y(O, O)))$$

Table 7. Proof of (10)

Number	Proof	Law
1	$\forall x \exists y(\psi(x) \to \varphi(y))$	Pr
2	$\forall x \forall y(\psi(x, y))$	Pr

DOI: 10.54254/2753-7064/2026.36988

Table 7. (continued)

3	$(\neg\exists x(I,O) \wedge (\exists y(I,I) \vee \exists y(O,I) \vee \exists y(O,O)))$ $\wedge((\neg\mathscr{S}x(I,O)) \rightarrow (\mathscr{S}y(I,I) \vee \mathscr{S}y(O,I) \vee \mathscr{S}y(O,O)))$	$\leftrightarrow$ E , (11), 1
4	$((\neg\exists x(O,I) \wedge \neg\exists x(O,O)) \wedge (\neg\exists y(O,I) \wedge \neg\exists y(O,O)))$ $\wedge((\neg\mathscr{S}x(O,I) \wedge \neg\mathscr{S}x(O,O)) \rightarrow (\neg\mathscr{S}y(O,I) \wedge \neg\mathscr{S}y(O,O)))$	$\leftrightarrow$ E , (12), 2
5	$\neg\exists x(I,O)$	$\wedge$E , 3
6	$\neg\exists x(O,O)$	$\wedge$E , 4
7	$\neg\exists x(I,O) \wedge \neg\exists x(O,O)$	$\wedge$I , 5, 6
8	$\exists y(I,I) \vee \exists y(O,I) \vee \exists y(O,O)$	$\wedge$E , 3
9	$\exists y(I,I) \vee \exists y(O,I)$	DS, 8, 6
10	$(\neg\exists x(I,O) \wedge \neg\exists x(O,O)) \wedge (\exists y(I,I) \vee \exists y(O,I))$	$\wedge$I , 7, 9
11	$\neg\mathscr{S}x(I,O) \wedge \neg\mathscr{S}x(O,O)$	Assumption
12	$\neg\mathscr{S}x(I,O)$	$\wedge$E , 11
13	$\neg\mathscr{S}x(O,O)$	$\wedge$E , 11
14	$(\neg\mathscr{S}x(I,O)) \rightarrow (\mathscr{S}y(I,I) \vee \mathscr{S}y(O,I) \vee \mathscr{S}y(O,O))$	$\wedge$E , 6
15	$\mathscr{S}y(I,I) \vee \mathscr{S}y(O,I) \vee \mathscr{S}y(O,O)$	$\rightarrow$ E , 14, 15
16	$\mathscr{S}y(I,I) \vee \mathscr{S}y(O,I)$	DS, 18, 16

DOI: 10.54254/2753-7064/2026.36988

Table 7. (continued)

17	$(\neg\mathscr{S}x(I,O)\wedge\neg\mathscr{S}x(O,O))\rightarrow(\mathscr{S}y(I,I)\vee\mathscr{S}y(O,I))$	$\rightarrow$ I , 14-19
18	$((\neg\exists x(I,O)\wedge\neg\exists x(O,O))\wedge(\exists y(I,I)\vee\exists y(O,I)))$ $\wedge((\neg\mathscr{S}x(I,O)\wedge\neg\mathscr{S}x(O,O))\rightarrow(\mathscr{S}y(I,I)\vee\mathscr{S}y(O,I)))$	$\wedge$I , 14, 20
19	$\forall x\exists y(\varphi(x,y))$	$\leftrightarrow$ E , 3, 21

$$\forall x(\psi(x))\leftrightarrow(\neg\exists x(O,I)\wedge\neg\exists x(O,O))\wedge(\neg\mathscr{S}x(O,I)\wedge\neg\mathscr{S}x(O,O)) \tag{16}$$

$$\exists y(\varphi(y))\leftrightarrow(\exists y(I,I)\wedge\exists y(O,I))\wedge(\mathscr{S}y(I,I)\wedge\mathscr{S}y(O,I)) \tag{17}$$

Thus, the following references can be given in Table 8.

Table 8. Proof of (14) in basic FOL

Number	Proof	Law
1	$\forall x\exists y(\psi(x)\rightarrow\varphi(y))$	Pr
2	$\forall x(\psi(x))$	Pr
3	$((\neg\exists x(I,O))\wedge(\exists y(I,I)\vee\exists y(O,I)\vee\exists y(O,O)))$ $\wedge((\neg\mathscr{S}x(I,O))\rightarrow(\mathscr{S}y(I,I)\vee\mathscr{S}y(O,I)\vee\mathscr{S}y(O,O)))$	$\leftrightarrow$ E , (15), 1
4	$(\neg\exists x(O,I)\wedge\neg\exists x(O,O))\wedge(\neg\mathscr{S}x(O,I)\wedge\neg\mathscr{S}x(O,O))$	$\leftrightarrow$ E , (16), 2
5	$(\neg\exists x(I,O))\wedge(\exists y(I,I)\vee\exists y(O,I)\vee\exists y(O,O))$	$\wedge$E , 3
6	$\neg\exists x(O,I)\wedge\neg\exists x(O,O)$	$\wedge$E , 4

DOI: 10.54254/2753-7064/2026.36988

Table 8. (continued)

7	$(\neg\mathscr{S}x(I,O)) \rightarrow (\mathscr{S}y(I,I) \vee \mathscr{S}y(O,I) \vee \mathscr{S}y(O,O))$	∧E , 3
8	$\neg\mathscr{S}x(O,I) \wedge \neg\mathscr{S}x(O,O)$	∧E , 4

From 5 and 6, $\exists y(I,I) \wedge \exists y(O,I)$ can't be derived; from 7 and 8, $\mathscr{S}y(I,I) \wedge \mathscr{S}y(O,I)$ can't be derived either. Therefore, the premises are consistent, but the inference is invalid.

This kind of case isn't rare. Therefore, not all inferences with premises and a conclusion in prenex normal form that are valid in classical FOL remain valid in basic FOL. However, this doesn't show that the method proposed in this paper is wrong. Only if at least one inference can be derived using the rules of basic FOL, but is invalid in classical FOL, can be considered a problem of the propositionalization method for basic FOL.

As with syllogisms, there are 15 valid rules in basic syllogistic logic and 24 valid rules in strengthened syllogistic logic, and all valid rules of basic syllogistic logic remain valid in strengthened syllogistic logic [9]. FOL has a similar relation: the valid rules of basic FOL should be included in the valid rules of strengthened FOL. However, any inference that is invalid in basic FOL but valid in strengthened FOL depends for its validity on the additional condition that the domain is non-empty.

4.6. A brief introduction to strengthened FOL

Since the focus of this paper is basic FOL, strengthened FOL can only be briefly introduced here. Strengthened FOL is constructed under the guidance of ODC. Its idea is consistent with quantifier expansion, but the direction of expression is reversed. For example, if all the elements in the domain that have a sentence $\forall x(\psi(x))$ are $a_1, a_2, \ldots, a_n$, then $\forall x(\psi(x))$ can be expanded as $\psi(a_1) \wedge \psi(a_2) \wedge \ldots \wedge \psi(a_n)$ [11]. However, ODC works in the opposite direction. $\psi(a_1) \wedge \psi(a_2) \wedge \ldots \wedge \psi(a_n)$ is concluded as $\forall$x(ψ(x)). However, the results of these two directions are the same; both lead to

$$\forall x(\psi(x)) \leftrightarrow \psi(a_1) \wedge \psi(a_2) \wedge \ldots \wedge \psi(a_n) \tag{18}$$

For existential propositions, the idea is that there is a $\psi(a_m)$ in the domain first, where a_m is an element of the domain, and then $\exists x(\psi(x))$ follows.

$$\psi(a_m) \rightarrow \exists x(\psi(x)) \tag{19}$$

However, once such a generalization is made, it is like a drop of water that flows into the sea and can no longer be found. Thus, it can only be

DOI: 10.54254/2753-7064/2026.36988

$$\exists x(\psi(x)) \leftrightarrow \psi(a_1) \vee \psi(a_2) \vee \ldots \vee \psi(a_n) \tag{20}$$

However, this method may face an important limitation. Since propositional formulas are finite in length, if propositionalization depends on representing the elements of the domain one by one, this method can only be applied to finite domains [11].

4.7. Advantages and disadvantages of basic FOL

4.7.1. The advantages of basic FOL

The advantage of basic FOL is that, within the range it can handle, the problem of infinity can be ignored. It doesn't concern itself with individual objects, but only with the regions to which they belong. It doesn't care about the objects, but only about their regions. That is, even if there are infinitely many objects, as long as the number of regions is finite, it is still useful. A predicate can have infinitely many objects in its extension, but in classical FOL, a sentence has finite length. Therefore, there are only finitely many predicates in a sentence. Therefore, within the range that basic FOL can handle, this method can be applied more widely than quantifier expansion, because it can propositionalize sentences with infinite domains.

4.7.2. The limitation of basic FOL

The propositionalization method proposed in this paper for basic FOL has one limitation: at present, it can only handle sentences in prenex normal form, but it can't handle other forms of FOL sentences. The reason for this limitation can't be explained yet. However, in a finite domain, this part can be supplemented with a strengthened FOL. However, strengthened FOL is also limited to finite domains but can't handle infinite domains. Therefore, non-prenex normal form sentences with infinite domains still can't be propositionalized by the two methods proposed in this paper for basic FOL and strengthened FOL.

4.8. Potential applications

The propositionalization method for basic FOL can be applied to propositionalize expressions in set theory and then used to test whether set-theoretic rules are valid. Therefore, it may have applications in mathematics and physics. Since the method is also decidable, it may further be applied in computer science.

For example, the addition $A + B$ and can be defined as

$$\forall x(((x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)) \leftrightarrow x \in A + B) \tag{21}$$

The multiplication of dimensions and can also be defined as

$$\forall x \forall y((x \in C \wedge y \in D) \leftrightarrow (x, y) \in C \times D) \tag{22}$$

DOI: 10.54254/2753-7064/2026.36988

5. Conclusion

This paper proposes two methods for propositionalizing FOL sentences based on different semantics, and it focuses on basic FOL, which allows an empty domain. Based on the Region Table, prenex normal form sentences can be translated into TFL sentences. Then, inferences formed from such sentences can be translated into TFL inferences. Therefore, their validity can be tested using TFL rules.

Compared with ordinary quantifier expansion, this method doesn't require objects in the domain to be enumerated one by one. Therefore, it avoids the limitation of a finite domain. Within its scope of application, the method can also handle infinite domains.

References

[1] Whitehead, A. N., & Russell, B. (1925). Principia mathematica: By Alfred North Whitehead... And Bertrand Russell... 2nd ed., Vol. 1. The University Press. https://link.gale.com/apps/doc/BCSLQZ838702469/NCCO?u=nysl_ce_syr&sid=gale_marc&xid=aa3838fb&pg=1

[2] Cohen, S. M., & Reeve, C. D. C. (2025). Aristotle's metaphysics. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford Encyclopedia of Philosophy* (Spring 2025 ed.). Metaphysics Research Lab, Stanford University. https://plato.stanford.edu/entries/aristotle-metaphysics/

[3] Smith, R. (2022). Aristotle's logic. In E. N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy*. https://plato.stanford.edu/entries/aristotle-logic/

[4] Open Logic Project. (2026). *The Open Logic Text*. https://builds.openlogicproject.org/open-logic-complete.pdf

[5] Byrne, A. (2025). Inverted qualia. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford Encyclopedia of Philosophy* (Winter 2025 ed.). Metaphysics Research Lab, Stanford University. https://plato.stanford.edu/entries/qualia-inverted/

[6] Caston, V. (2026). Concepts in classical Greece: Aristotle and his predecessors. In H. Taieb & S. Schmid (Eds.), *A philosophical history of the concept* (pp. 16–44). Cambridge University Press. https://doi.org/10.1017/9781009273831.002

[7] Lee, A. Y. (2024). Objective phenomenology. *Erkenntnis, 89*(3), 1197–1216. https://doi.org/10.1007/s10670-022-00576-0

[8] Tse, L., & Legge, J. (2009). *Tao Te Ching: Or the Tao and its characteristics* (1st ed.). The Floating Press. https://ia902903.us.archive.org/20/items/taotechingjmaesleggefrominternet_454_o/Tao%20Te%20Ching%20Jmaes%20Legge%20%28From%20Internet%29.pdf

[9] Chen, X. (2006). *Ziran yanyi luoji daolun [Introduction to natural deduction logic]* (2nd ed.). Sun Yat-sen University Press. https://www.scribd.com/document/930403234/%E8%87%AA%E7%84%B6%E6%BC%94%E7%BB%8E%E9%80%BB%E8%BE%91%E5%AF%BC%E8%AE%BA

[10] Boole, G. (1847). *The mathematical analysis of logic: Being an essay towards a calculus of deductive reasoning*. Macmillan, Barclay, & Macmillan. https://www.gutenberg.org/files/36884/36884-pdf.pdf

[11] Paseau, A. C. (2021). Propositionalism. *The Journal of Philosophy, 118*(8), 430–449. https://doi.org/10.5840/jphil2021118830